Mitigating the Effect of Noise in Iterative Projection Phase Retrieval

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Abstract: Here the effect of noisy measurement data is explored within the traditional phase retrieval problem with the goal of filtering the noise to obtain an estimate of the true data. The method proposed can be applied to most existing phase retrieval methods.
OCIS codes: (100.3020) Image reconstruction-restoration; (100.5070) Phase retrieval.

1. Introduction

Since the renowned papers [1, 2] by Fienup, many methods of iterative phase retrieval have been devised that address the problem where a two dimensional image is the object of interest and only the magnitude of its Fourier transform is known. Thus the phase of its Fourier transform needs to be estimated to recover the image. Building upon Gerchberg and Saxton's Error Reduction method [3], Fienup's Hybrid Input-Output (HIO) method serves as the basis for comparison of most other methods. It projects the estimate of the image from the image domain to the Fourier domain repeatedly and imposes various constraints upon each projection in an attempt to find the domains' intersection. While the only viable method, the projective approach has a major caveat: it is susceptible to stagnation. Explicitly, the error metric contains local minima which cause the minimization process to halt before reaching the global minimum. Much research after the development of the HIO method has focused on modifying the constraints imposed with the goal of finding the global minimum. Not much attention, however, has been placed on filtering noise in the modulus data.

Directly addressing the issue of noise in the Fourier modulus data introduces a change to the problem statement which has a severe effect, as will be shown later. Where the ideal phase retrieval problem statement is to find the intersection of the Fourier and image domains via successive projections, noise in the modulus data may cause the two domains not to intersect. The problem statement must be reformulated such that the minimum distance between the domains is sought, not the intersection. The examples shown here reveal that not having an intersection can cause many phase retrieval methods to diverge. This discussion expands on the work in [4] and includes applications to phase retrieval methods beyond the HIO.

The formal problem statement here is to devise a projective phase retrieval algorithm capable of filtering noise from the Fourier modulus data and converging to the true image. This expands on some previous attention to noise in the modulus data [5, 6] in that not only is the algorithm required not to diverge, the noise must be filtered. Some papers have already approached this problem from the viewpoint of mitigating divergence at high noise levels; however, they did not have the end goal of filtering the noise. Comparisons will be presented to these methods.

A useful compilation of some of the best phase retrieval methods was published by Marchesini [7]. He performed a side-by-side comparison of some phase retrieval methods in a simple two degree-of-freedom example which will be used extensively in the work here. The example problem consists of finding the intersection between two domains in two dimensional space. This example allows for a visualization of the projections and constraints at each iteration.

2. Effects and Filtering of Noise

The various projections in the phase retrieval algorithms typically follow a notation using projection operators. Two main operators are thus defined. The modulus domain projection is

$$P_m = FT^{-1}\Big[\big|F(u,\mathrm{v})\big|\exp\Big[i\arg\Big(FT\big[g(x,y)\big]\Big)\Big]\Big]. \tag{1}$$

$FT[\cdot]$ is the discrete Fourier transform, and $|F(u,v)|$ is the measured modulus. The image domain projection is

$$P_s = \begin{cases} 1, & (x,y)\in\gamma \\ 0, & \text{otherwise} \end{cases} \tag{2}$$

where γ is the foreground region. Each phase retrieval algorithm has its own use of these projections, but the simplest is the Error Reduction method which is $g_{k+1}(x,y) = P_s P_m g_k(x,y)$.
where $g_k(x,y)$ is the image estimate at the k^{th} iteration.

As shown previously in [4], the noise in the modulus data can be described as a deviation about the true value as in

$$\tilde{G}(u,v) = F_{True}(u,v) + \sum_{(a,b)} \Delta(a,b)\,\delta(u-a,v-b) \tag{3}$$

where Δ is the magnitude of the noise at the (a,b) location in the (u,v) plane. When exploiting the linearity of the Fourier transform and using the support constraint of the Error Reduction method, each of the summed terms in (3) can be analyzed separately. Following the steps of the Error Reduction projections, the constrained Fourier modulus for the noise term is

$$G'(u,v)=\Delta\delta(u-a,v-b) \tag{4}$$

It follows that the unconstrained Fourier modulus is

$$|G(u,v)|=\frac{\Delta}{MN}\left[\sin\left(\frac{M-2A}{M}\pi u\right)\sin\left(\frac{N-2B}{N}\pi v\right)\right]\left[\sin\left(\frac{1}{M}\pi u\right)\sin\left(\frac{1}{N}\pi v\right)\right]^{-1}. \tag{5}$$

after going through the image domain projection. The maximum and squared Frobenius norms of (5) are both decreased by the factor

$$(M-2A)(N-2B)M^{-1}N^{-1} \tag{6}$$

when compared to the modulus before the iteration as expressed in (4). This leads to the idea that the Fourier modulus constraint should preserve some of the data in (5) and not completely constrain the modulus to be equal to (4).

It is proposed that the Fourier domain projection should be relaxed with the parameter λ in the form

$$P_m=FT^{-1}\left[\lambda FT\left[g_k(x,y)\right]+(1-\lambda)|F(u,v)|\exp\left(i\arg\left(FT\left[g_k(x,y)\right]\right)\right)\right]. \tag{7}$$

This projection constrains the modulus to stay near the measured quantity but allows deviation based on the previous iteration's Fourier domain. If λ is large enough, the Fourier domain will deviate enough to create an intersection with the image domain where an intersection previously may not have existed. The value of λ should be increased cautiously as not to create an intersection near a local minimum instead of the global minimum.

3. Examples

As an example, consider the image and Fourier domains in two dimensional space. The Fourier domain M is non-linear and does not intersect the image support domain S. The algorithm will seek the point of minimum separation of the two domains shown in Figures 2a and 2b. Using the traditional Fourier domain projection, the various algorithms diverge as shown in Figure 1a. The algorithms shown are the Error Reduction [3], Solvent Flipping [8], Hybrid Input-Output [2], Difference Map [9], Averaged Successive Reflections [10], Hybrid Projection Reflections [11], Random Averaged Alternating Reflector [12], and the Levi-Stark method [13]. Shown in Figure 1b, using the new Fourier domain projection causes the solutions to converge to the point where the two domains have the minimum separation except for the error reduction and solvent flipping methods which stagnate at the local minimum. Further comparisons within the two dimensional example will be shown in the full manuscript revealing more complex behaviors.

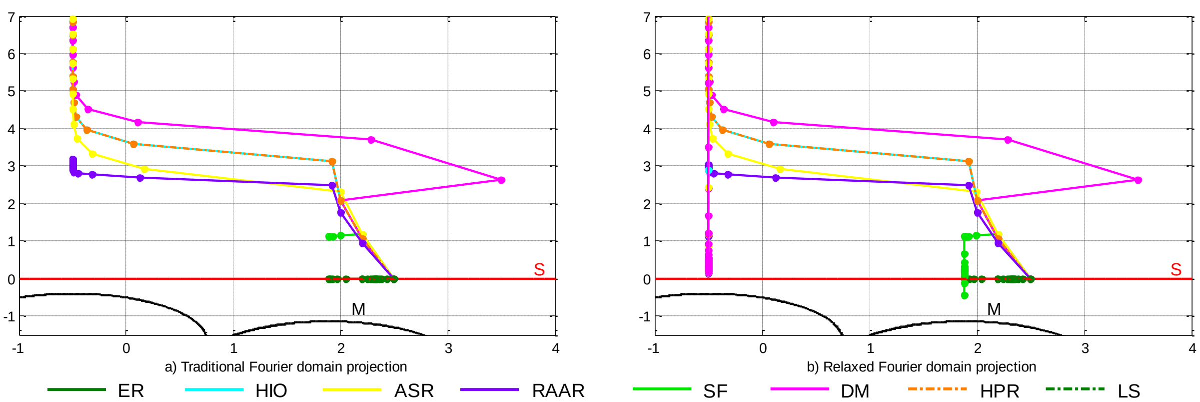


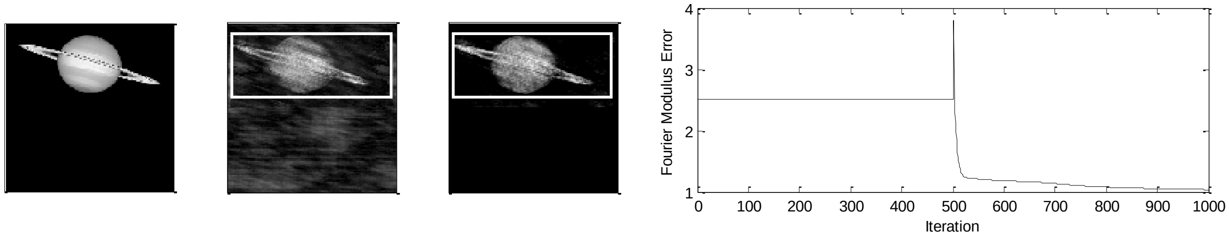
Figure 1: Comparison of various phase retrieval projection methods using the a) traditional and b) new Fourier domain projection.

As a practical example, consider the photograph of Saturn as shown in Figure 2a. The image has a fairly large amount of oversampling which means the filtering effect should be relatively large when the relaxation is implemented due to the effect shown in equation (6). For the first 500 iterations the relaxation parameter is zero and the standard HIO is implemented with the feedback parameter set to 0.9 as is typical. This results in a noisy, blurry image as shown in Figure 2b. After iteration 500 the relaxation parameter was set to 0.9. The background region of the image, both outside and within γ, is nearly devoid of artifacts. The foreground also shows some improvement; however, the image is suffering from a convolution with a flipped image. Using $|G'_{100}(u,v)|$ as the Fourier modulus instead of $|F(u,v)|$, algorithms such as those presented in [14, 15] can satisfactorily deconvolute the multiple solutions.

The squared error in the Fourier modulus data at the k^{th} iteration can be defined as

$$E_k^2 = N^{-2} \sum_{(u,v)} \Big[\left| G_k(u,v) \right| - \left| F(u,v) \right| \Big]^2. \tag{8}$$

Figure 2d shows the typical trend where the error is constant until the relaxation occurs. Upon λ being set to a non-zero value, a spike occurs followed by a marked decrease. This result definitively shows that the noise is being filtered from the modulus data.



a) True Image b) Estimate, Iteration 500 c) Estimate, Iteration 1000 d) Fourier modulus error vs. iterations.

Figure 2: Example showing (a) the true image, (b) the reconstructed image after 500 iterations of the HIO, (c) the reconstructed image after an additional 500 iterations using the HIO with the new modulus projection, and (d) the Fourier modulus error at each iteration. The box in (b) and (c) indicates the boundary of γ .

4. Conclusion

The work summarized here shows the ability of any projective phase retrieval algorithm to improve its performance in the presence of noise with the addition of a relaxed Fourier domain projection. The traditional projection rigidly imposes the measured data; however, here the projection combines the information within the measured and the previous iteration's modulus, which contains influence from the image domain projection. The result is a decrease in the amount of noise present in the data after very few iterations. Trials have been run which show up to a 65% reduction in the error metric in equation (8). The full manuscript will contain comparisons to many phase retrieval methods and characterization of the behavior parameterized by the amount of oversampling and the relaxation parameter.

5. References

[1] J. R. Fienup, "Reconstruction of an object from the modulus of its Fourier transform," *Optics Letters,* vol. 3, no. 1, pp. 27-29, 1978.

[2] J. R. Fienup, "Phase Retrieval Algorithms: A Comparison," *Applied Optics,* vol. 21, no. 15, pp. 2758-2769, 1 Aug 1982.

[3] R. W. Gerchberg and W. O. Saxton, "A practical algorithm for the determination of the phase from image and diffraction plane pictures," *Optik,* vol. 35, no. 237, 1972.

[4] R. Trahan and D. Hyland, "Mitigating the effect of noise in the hybrid input–output method of phase retrieval," *Applied Optics,* vol. 52, no. 13, pp. 3031-3037, 2013.

[5] G. Liu, "Object reconstruction from noisy holograms: multiplicative noise model," *Optics Communications,* vol. 79, no. 6, pp. 402-406, 1990.

[6] M. Kohl, A. A. Minkevich and T. Baumback, "Improved success rate and stability for phase retrieval by including randomized overrelaxation in the hybrid input-output algorithm," *Opt. Express,* vol. 20, pp. 17093-17106, 2012.

[7] S. Marchesini, H. He, H. N. Chapman, S. P. Hau-Riege, A. Noy, M. R. Howells, U. Weierstall and J. Spence, "X-ray image reconstruction from a diffraction pattern alone," *Physical Review B,* vol. 68, no. 140101, 2003.

[8] J. P. Abrahams and A. W. G. Leslie, "Methods used in the structure determination of bovine mitochondrial F1 ATPase," *Acta Crystallogr. D Biol. Crystallogr.,* vol. 52, pp. 30-42, 1996.

[9] Elser, Viet, "Phase retrieval by iterated projections," *J. Opt. Soc. Am. A,* vol. 20, no. 1, pp. 40-55, 2003.

[10] H. H. Bauschke, P. L. Combettes and D. R. Luke, "Phase retrieval, error reduction algorithm, and Fienup variants: a view from convex optimization," *J. Opt. Soc. Am. A,* vol. 19, no. 7, pp. 1334-1345, 2002.

[11] H. H. Bauschke, P. L. Combettes and D. R. Luke, "Hybrid projection–reflection method for phase retrieval," *J. Opt. Soc. Am. A,* vol. 20, no. 6, pp. 1025-1034, 2003.

[12] D. R. Luke, "Relaxed averaged alternating relfections for diffraction imaging," *Inverse Problems,* vol. 21, pp. 37-50, 2005.

[13] A. Levi and H. Stark, "Image restoration by the method of generalized projections with application to restoration from magnitude," *J. Opt. Soc. Am. A,* vol. 1, pp. 932-943, 1984.

[14] J. R. Fienup and C. C. Wackerman, "Phase-retrieval stagnation problems and solutions," *J. Opt. Soc. Am. A,* vol. 3, no. 11, pp. 1897-1907, 1986.

[15] J. Zhao, D. Want, F. Zhang and Y. Want, "Hybrid phase retrieval approach for reconstruction of in-line digital holograms without twin images," *Opt. Eng.,* vol. 50, no. 9, 2011.